MTH 44307

3 UNIT

COURSE OUTLINE

TOPICS	OUTLINE	DURATION
POLYNOMIALS	• SPLINE • ORTHOGONAL • CHEBYSHEV • RECURRENCE RALATION	2 WEEKS
DIRECT $ INDIRECT METHODS FOR SOLUTION OF SYSTEM OF LINEAR EQUATIONS	• EULER'S METHOD (EM) • DTM METHOD (DTM) • RUNGE-KUTTA RK4 (RK4) • NEW ITERATIVE METHODS (NIM)	3 WEEKS
EIGEN VALUE PROBLEM	• POWER METHODS • INVERSE POWER METHOD	2 WEEKS
BOUNDARY VALUE PROBLEMS BVP $ PARTIAL DIFFERENTIAL PROBLEMS	• FINITE METHODS	2 WEEKS
SOFTWARE PRACTICAL LAB.	• MAPLE $ MATLAB	2 WEEKS
TEST	• TEST	1 WEEK

1.0 POLYNOMIAL AND SPLINE APPROXIMATION.

1.1 DEFINITION.

A polynomial is a function that can be written in this form

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + --- + a_nx^n \quad --- \quad (1.1.1)$$

for some coefficient $a_0, a_1, a_2, ---, a_n$ if $a_n \neq 0$, then the polynomial is said to be of order n.

A first order (linear) polynomial is just the equation of a straight line while a second order (Quadratic) polynomial describes a parabola.

Polynomials are just about the simplest mathematical functions that exist requiring only multiplication and addition for their evaluation. Yet, they are also having flexibility to represent very general non-linear relation-ship. Approximation of a more complicated function by polynomial is a basic building blocks for a great many numerical technique.

There are two (2) distinct purposes to which polynomial approximation is put in numerical analysis

(i) To model a non-linear relationship between a response variable and explanatory variable. The response variable is usually measured with and the interest is on the shape of the fitted curve and perhaps, also on the fitted polynomial coefficients.

(ii) To approximate a difficulty, to evaluate a function such as a density or a distribution function.

1.2 RECURRENCE RELATION.

A recurrence relation is a relation which can be used to generate terms of a sequence from some previous terms of the sequence.

The order of a recurrence relation is the numbers of the previous terms that are needed in other to compute the next terms of the sequence. thus, a general recurrence relation of order N is an equation of the form

$$U_{K+N} = f(U_{K+N-1}, U_{K+N-2}, U_{K+N-3}, ---, U_K) \quad ---- \quad (1.2.1)$$

The recurrence relation (1.2.1) is said to be a linear recurrence if f is a linear function of the previous terms. $(K \geq 0)$

$$U_{K+N} = C - a_{N-1}U_{K+N-1} - a_{N-2}U_{K+N-2} - ---- - a_1U_{K+1} - a_0U_K \quad --- \quad (1.2.2)$$

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e generate members of the sequence from the suitable initial values using the recurrence relation and compute the ratio

$$q_K = \frac{u_{K+1}}{u_K} \quad ---- \quad (1.4.2)$$

provided the initial values are chosen such that $A_1 \neq 0$, it follows that $q_K \to \alpha_1$ as $\alpha_1 \to \infty$. To find the smallest root (solution), we apply the same techniques to the polynomial

$$a_N + a_{N-1}u_{K+N-1} + a_{N-2}u_{K+N-2} + a_{N-3}u_{K+N-3} + --- + a_1 u_{K+1} + a_0 u_K = 0$$

$$\equiv a_N + a_{N-1}x + a_{N-2}x^2 + --- + a_1 x^{N-1} + a_0 x^N = 0.$$

whose roots are the reciprocal of those of the original equation (1.3.1)

EXAMPLE 1.

Generate the next five terms of the sequence defined by the recurrence relation

$$15u_{K+3} + 14u_{K+2} - 7u_{K+1} - 6u_K = 0.$$

where $u_0 = u_1 = 0$ and $u_2 = 1$

Solution.

$$15u_{K+3} = 6u_K + 7u_{K+1} - 14u_{K+2}$$

$$u_{K+3} = \frac{6u_K + 7u_{K+1} - 14u_{K+2}}{15}$$

Let $K = 0$

$$u_3 = \frac{6(0) + 7(0) - 14(1)}{15} = \frac{-14}{15}$$

when $K = 1$

$$u_4 = \frac{6u_1 + 7u_2 - 14u_3}{15} = \frac{6(0) + 7(1) - 14\left(\frac{-14}{15}\right)}{15} = \frac{301}{225} = 1.33778$$

when $K = 2$.

$$u_5 = \frac{6u_2 + 7u_3 - 14u_4}{15} = \frac{6(1) + 7\left(-\frac{14}{15}\right) - 14(1.33778)}{15} = -1.28415.$$

when $K = 3$

$$u_6 = \frac{6u_3 + 7u_4 - 14u_5}{15} = \frac{6\left(-\frac{14}{15}\right) + 7(1.33778) - 14(-1.28415)}{15} = 1.44950$$

$$u_7 = \frac{6u_4 + 7u_5 - 14u_6}{15} = \frac{6(1.33778) + 7(-1.28415) - 14(1.44950)}{15} = -1.41703.$$

equation (1.2.2) is said to be homogenous if the Constant $C=0$, in that case, the relation could be written as

$$u_{k+N} + a_{N-1}u_{k+N-1} + a_{N-2}u_{k+N-2} + \cdots + a_1 u_{k+1} + a_0 u_k = 0 \quad \text{----} \quad (1.2.3)$$

The further modification of (1.2.3) is allowing a general non-zero leading coefficient will make the connection with polynomial equation more apparent. We then have

$$a_N u_{k+N} + a_{N-1}u_{k+N-1} + a_{N-2}u_{k+N-2} + \cdots + a_1 u_{k+1} + a_0 u_k = 0 \quad \text{-----} \quad (1.2.4)$$

In other to generate terms using any of this recurrence relation will require some initial value. For a recurrence relation of order N there must be a sequence of order N in the initial value such as $u_0, u_1, u_2, u_3, \ldots, u_{N-1}$ is required.

1.3 EQUIVALENCE BETWEEN POLYNOMIALS AND RECURRENCE RELATION.

The recurrence relation (1.2.4) is directly related to the polynomial of equation (1.1.1)

$$a_N x^N + a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \cdots + a_1 x + a_0 = 0 \quad \text{---} \quad (1.3.1)$$

In the sense of finding all solutions (1.3.1) is equivalent to finding the general solution of (1.2.4).
In the simplest case, where all the solutions of (1.3.1) are distinct, the general solution of (1.2.4) is given by

$$u_k = A_1\alpha_1^k + A_2\alpha_2^k + A_3\alpha_3^k + \cdots + A_n\alpha_n^k \quad \text{---} \quad (1.3.2)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the solutions of the polynomial equations (1.3.2) and $A_1, A_2, A_3, \ldots, A_n$ are arbitrary constants whose values may be determined from the set of the initial values.

1.4 BERNOULLI'S METHOD.

This method is the application of the equivalence between polynomials and recurrence relation to find the largest or the smallest root of a polynomial equation.
In this context, "largest" or "smallest" refers to the largest or smallest root in the absolute.
We assume that the largest root is isolated in absolute value and is denoted by

$$|\alpha_1| > |\alpha_i| \ \forall \ i > 1 \quad \text{----} \quad (1.4.1)$$

Provided that the coefficient A_1 in (1.3.2) is non-zero, then the ratio of the successful numbers of the sequence the will converge to α_1, thus,

1.5 BAIRSTOW'S METHOD.

This is a method that we use to solve polynomials. It is an iterative technique for finding a quadratic factor of a polynomial, thus, we seek a quadratic

$x^2 - ux - v$ which is a factor of a polynomial.

$$P(x) = a_N x^N + a_{N-1} x^{N-1} + a_{N-2} x^{N-2} + \cdots + a_1 x + a_0 \quad \text{----} \quad (1.5.1)$$

Using the remainder theorem, we can write

$$P(x) = (x^2 - ux - v) q(x) + cx + d \quad \text{----} \quad (1.5.2)$$

where q is a polynomial of degree $N-2$ and $(x^2 - ux - v)$ is a factor of P if $c = d = 0$

For conveniency in the algorithm, we write the remainder in the form $b_1(x-u) + b_0$ and the polynomial is

$$q(x) = b_N x^{N-2} + b_{N-1} x^{N-3} + b_{N-2} x^{N-4} + \cdots + b_3 x + b_2 \quad \text{----} \quad (1.5.3)$$

Thus

$$P(x) = (x^2 - ux - v)(b_N x^{N-2} + b_{N-1} x^{N-3} + \cdots + b_3 x + b_2) + (b_1(x - v) + b_0) \quad \text{---} \quad (1.5.4)$$

And seek the coefficient u, v, $\ni$ $b_1 = b_0 = 0$. Setting $b_{N+2} = b_{N+1} = 0$ and comparing the coefficient in equation (1.5.4), we obtain

$$b_k = a_k + u b_{k+1} + v b_{k+2} \quad \text{----} \quad (1.5.5)$$

$(k = N, N-1, N-2, \cdots, 1, 0)$ from (1.5.5)

we see immediately that the coefficient of b_k are all functions of u & v.

Thus, the equation to solve is now

$$b_1(u,v) = b_0(u,v) = 0 \quad \text{---} \quad (1.5.6)$$

To apply the method: This is an application of Newton law to the solution of (1.5.6).

In other to apply the Newton law, we require the Jacobian matrix for this system. The element of the matrix are derived using a similar recurrence relation to (1.5-5) with

$$c_k = \frac{\delta b_{k-1}}{\delta u} = \frac{\delta b_{k-2}}{\delta v} \quad \text{----} \quad (1.5.7)$$

The corresponding recurrence relation is

$$c_k = b_k + u c_{k+1} + v c_{k+2} \quad \text{---} \quad (1.5.8)$$

The desired inverse Jacobian is then given as

$$J^{-1} = \frac{1}{c_2^2 - c_1 c_3} \begin{bmatrix} c_2 & -c_3 \\ -c_1 & c_2 \end{bmatrix} \quad \text{----} \quad (1.5.9)$$

Thus, this can be used to generate an iterative as follows.

ALGORITHM.

Input: coefficients $a_0, a_1, a_2, ---, a_N$ of the polynomial's initial guess u, v for the coefficient of a quadratic factor.

Initialize

$$b_{N+1} = b_{N+2} = C_{N+1} = C_{N+2} = 0$$

Compute and repeat

$$u_0 = u_1, \quad v_0 = v_1$$

for $K = N$ down to 0.

$$b_K = a_K + u_0 b_{K+1} + v_0 b_{K+2}$$

for $K = N$ down to 1

$$C_K = b_K + u_0 C_{K+1} + v_0 C_{K+2}$$

$$\Delta = C_2^2 - C_1 C_3$$

$$u_1 = u_0 - (C_2 b_1 - C_3 b_0)/\Delta$$

$$v_1 = v_0 - (C_2 b_0 - C_1 b_1)/\Delta$$

until it converges.

Output: Quadratic function $(x^2 - ux - v)$. The convergence criteria could be based on the objective $b_1 = b_0 = 0$ or on the change in u, v.

EXAMPLE

Derive the iteration formula for the coefficient in Bairstow's method

Solution.

The required equation is: $b_1(u,v) = b_0(u,v) = 0$

Using Newton's iterative method is given by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - J^{-1}\begin{pmatrix} b_1 \\ b_0 \end{pmatrix}$$ where J is a Jacobian matrix of the system

$$J = \begin{vmatrix} \frac{\partial b_1}{\partial u} & \frac{\partial b_1}{\partial v} \\ \frac{\partial b_0}{\partial u} & \frac{\partial b_0}{\partial v} \end{vmatrix} = \begin{vmatrix} C_2 & C_3 \\ C_1 & C_2 \end{vmatrix} = C_2^2 - C_1 C_3 \text{ (from (1.5-7))}.$$

$$J = \begin{pmatrix} C_2 & C_3 \\ C_1 & C_2 \end{pmatrix} \quad ; \quad J_{cof} = \begin{pmatrix} C_2 & -C_1 \\ -C_3 & C_2 \end{pmatrix} \quad ; \quad J^T = \begin{pmatrix} C_2 & -C_3 \\ -C_1 & C_2 \end{pmatrix}$$

$$J^{-1} = \frac{1}{C_2^2 - C_1 C_3}\begin{pmatrix} C_2 & -C_3 \\ -C_1 & C_2 \end{pmatrix}$$

Thus

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \frac{1}{C_2^2 - C_1 C_3}\left(\begin{pmatrix} C_2 & -C_3 \\ -C_1 & C_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_0 \end{pmatrix}\right) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} - \frac{1}{\Delta}\begin{pmatrix} C_2 & -C_3 \\ -C_1 & C_2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_0 \end{pmatrix}$$

Finally, iterative of Bairstow's method is

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{bmatrix} u_0 - (C_2 b_1 - C_3 b_0)/\Delta \\ v_0 - (C_2 b_0 - C_1 b_1)/\Delta \end{bmatrix}$$

2.0 SPLINE INTERPOLATION.

2.1 DEFINITION.

A function S is a spline of degree β with knots $x_0 < x_1 < x_2 < \dots < x_n$ if
(a) on each subintervals (x_i, x_{i+1}) S is a polynomial of degree at most β, and
(b) $S, S', S'', \dots, S^{\beta-1}$ are all continuous at the knots.

NOTE: A spline S is necessary continuous on the interior of any of the subintervals and so the second condition (b) is equivalent to say that $S \in C^{\beta-1}[x_0, x_1]$ ---- (2.1.1)
so S is $(\beta-1)$ times continuously differentiable on $[x_0, x_n]$.
Thus, a spline of degree β is a piecewise polynomial of degree β with $(\beta-1)$ continuous derivatives, thus, a spline

$$S(x) = S_i(x) : x \in [x_{i-1}, x_{i+1}] \quad \text{------} (2.1.2)$$

Each component S_i is a polynomial of degree at most β.

2.2 LINEAR SPLINE.

A linear spline taking values $f_0, f_1, f_2, \dots, f_n$ at the knots $x_0 < x_1 < x_2 < \dots < x_n$ is a piecewise function while graph connects the points $(x_0, f_0), (x_1, f_1), \dots, (x_n, f_n)$ with a straight line segments. Its component are given by

$$S_i(x) = \frac{f_i + (x - x_i)(f_{i+1} - f_i)}{(x_{i+1} - x_i)} \quad \text{------} (2.2.1)$$

Which can be rearranged in the symmetric form

$$S_i(x) = \frac{f_{i+1}(x - x_i)}{(x_{i+1} - x_i)} - \frac{f_i(x - x_{i+1})}{(x_{i+1} - x_i)} \quad \text{------} (2.2.2)$$

In other words, a linear spline is a continuous function formed by connecting linear segment. The point where the segment connect are called the KNOTS of the spline.
A spline of degree β is a function formed by connecting polynomials segment of degree β so that
a) the function is continuous
b) the β^{th} derivative is constant
between the knots of the equation from a spline of degree β with K knots is

$$S_i = \beta + \sum_{d=1}^{\beta} B x^d + \sum_{k=1}^{K} b_K (x - \xi_1)_+^{\beta} \quad \text{------} (2.2.3)$$

The matrix for a spline of degree β with knots is the n by $1+\beta+K$ with entries

$$S_i = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{\beta} & (x_1 - \xi_1)_+^{\beta} & \cdots & (x_1 - \xi_K)_+^{\beta} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{\beta} & (x_2 - \xi_1)_+^{\beta} & \cdots & (x_2 - \xi_K)_+^{\beta} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{\beta} & (x_3 - \xi_1)_+^{\beta} & \cdots & (x_3 - \xi_{K})_+^{\beta} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{\beta} & (x_n - \xi_1)_+^{\beta} & \cdots & (x_n - \xi_K)_+^{\beta} \end{bmatrix}$$

·0 ORTHOGONAL POLYNOMIAL. *

e simplest possible system of linear equation has a diagonal coefficient matrix.
e normal equation can be simplified to this form by using a basis consisting
f orthogonal polynomials. The coefficients are then obtained by simple division.
The inner (dot) product (or L inner product) of two functions f, g over a closed interval [a,b] is defined by

$$(f,g) = \int_a^b f(x)g(x)\,dx \quad \text{------} \quad (3.1)$$

Note that this inner product is related to L_2 norm of f on [a,b]

$$\|f\|_2^3 \simeq (f,f) = \int_a^b [f(x)]^2\,dx \quad \text{----} \quad (3.2)$$

Two functions f and g are said to be orthogonal with respect to inner product (3.1) if

$$(f,g) = 0 \quad \text{----} \quad (3.3)$$

A system of orthogonal polynomial on [a,b] consists of polynomial ϕ_i of degree i (i = 0, 1, 2, ---) which are mutually orthogonal to

$$(\phi_i, \phi_j) = \int_a^b \phi_i(x)\phi_j(x)\,dx \quad \text{----} \quad (3.4)$$

The orthogonal polynomial $\phi_0, \phi_1, ---, \phi_N$ form a basis for the space π_N of polynomials of degree at most N. The normal equation for least square approximation to f using this basis, so that

$$P(x) = \sum_{i=1}^{N} a_i \phi_i(x) \quad \text{----} \quad (3.5)$$

where

$$a_i = \frac{(f_s, \phi_i)}{\|\phi_i\|_2^3} = \frac{\int_a^b f(x)\phi_i(x)\,dx}{\int_a^b [\phi_i(x)]^2\,dx} \quad \text{----} \quad (3.6)$$

The problem of least square polynomial approximation is thus reduced to find the appropriate system of orthogonal polynomials and integrating the function against these polynomials.
Finding such a system of orthogonal polynomials over one standard interval such as [-1,1] is sufficient since if two functions f, g are orthogonal over [-1,1] then the functions F, G given by

$$F(x) = f\left(\frac{2x-(a+b)}{b-a}\right) \quad \text{----} \quad (3.7)$$

$$G(x) = g\left(\frac{2x-(a+b)}{b-a}\right) \quad \text{----} \quad (3.8)$$

are orthogonal on [a,b]

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PROPERTIES OF ORTHOGONAL POLYNOMIALS. *

ny system of orthogonal matrix, that is the leading coefficient is 1 satisfies for recurrence relation

$$\phi_n(x) = (x - a_n)\phi_{n-1} - b_n\phi_{n-1} \quad \text{———} (3.9)$$

with $\phi_1 = 0$, $\phi_0 = 1$ and the coefficients are given by.

$$a_n = \frac{(x\phi_{n-1}, \phi_{n-1})}{\|\phi_{n-1}\|^2} \quad \text{———} (3.10)$$

$$b_n = \frac{(x\phi_{n-1}, \phi_{n-2})}{\|\phi_{n-2}\|^2} \quad \text{———} (3.11)$$

pproximation using monic orthogonal polynomials
e approximating polynomial particularly make evaluation of straightforward.

Let $P(x) = \sum_{i=1}^{N} \alpha_i \phi_i(x)$ where the basis consists of monic thogonal polynomial satisfying (3.10) and (3.11) using equation (3.9) e obtain table 1 below

ϕ_0 ; $i \geq 0$	Corresponding orthogonality condition
ϕ_0	1
ϕ_1	$x - a$
ϕ_2	$x^2 - a_2x + b_2$ $x^3 - a_2x^2 + b_2x$
ϕ_3	$x^3 - a_3x^2 + b_3x - c_3$ $x^4 - a_3x^3 + b_3x^2 - c_3x$ $x^5 - a_3x^4 + b_3x^3 - c_3x^2$
ϕ_4	$x^4 - a_4x^3 + b_4x^2 - c_4x + d_4$. $x^5 - a_4x^4 + b_4x^3 - c_4x^2 + d_4x$ $x^6 - a_4x^5 + b_4x^4 - c_4x^3 + d_4x^2$ $x^7 - a_4x^6 + b_4x^5 - c_4x^4 + d_4x^3$

and

$$b_2(x) = \frac{\int_0^1 (x\phi_1, \phi_0) w}{\int_0^1 \|\phi_0\|^2 w} \qquad \phi_0 = 1 \text{ and } \phi_1 = x - 3/5 .$$

$$b_2(x) = \frac{\int_0^1 x(x-3/5)\sqrt{x}\,dx}{\int_0^1 \sqrt{x}\,dx} = \frac{\int_0^1 x^{3/2}(x-3/5)\,dx}{\int_0^1 x^{1/2}\,dx}$$

$$= \frac{\int_0^1 (x^{5/2} - 3/5\, x^{3/2})\,dx}{\int_0^1 x^{1/2}\,dx} = \frac{2/7\, x^{7/2} - 6/25\, x^{5/2}\Big|_0^1}{2/3\, x^{3/2}\Big|_0^1} .$$

$$= \frac{\frac{2}{7} - \frac{6}{25}}{2/3} = \frac{8/175}{2/3} = \frac{12}{175}$$

Thus,

$$\phi_2(x) = x^2 - a_2 x + b_2$$

$$\phi_n = (x - a_n)\phi_{n-1} - b_n \phi_{n-1}$$

$$\phi_2 = (x - a_2)\phi_1 - b_2 \phi_1$$

$$\therefore \quad \phi_2 = \left(x - \frac{23}{45}\right)\left(x - 3/5\right) - \left(\frac{12}{175}(x - 3/5)\right)$$

$$\equiv x^2 - a_2 x + b_2 .$$

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om (ii): $b_3 = \frac{-9+10a_3}{10}$ --- (iii)

Substitute for b_3 in (i)

$$-16a_3 + 15\left(\frac{-9+10a_3}{10}\right) = -15$$

$$-160a_3 - 135 + 150a_3 = -150$$

$$-10a_3 = -15$$

$$a_3 = {}^{15}/_{10} = {}^{3}/_{2}$$

Substitute for a_3 in (iii)

$$b_3 = \frac{-9+10(3/2)}{10} = \frac{-9+15}{10}$$

$$b_3 = {}^{6}/_{10} = {}^{3}/_{5}$$

Substitute for a_3 and b_3 in

$$c_3 = -\frac{a_3}{3} + \frac{b_3}{2} + \frac{1}{4}$$

$$c_3 = -\frac{3}{6} + \frac{3}{10} + \frac{1}{4} = \frac{1}{20}$$

$$\therefore \phi_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.$$

XAMPLE 2.

nd the first 3 orthogonal polynomial using weighted function $\sqrt{x}$ on the interval
,1]

Solution.

TE: Using weighted function does not imply using table lout to show $\phi_0(x)$, (x), $\phi_2(x)$. Note that $\phi_0(x) = 1$

$$a_n = \frac{\int_0^1 (x\phi_{n-1}, \phi_{n-1})w}{\int_0^1 \|\phi_{n-1}\|^2 w} \quad ; \quad b_n = \frac{\int_0^1 (x\phi_{n-1}, \phi_{n-2})w}{\int_0^1 \|\phi_{n-2}\|^2 w}$$

Since $\phi_0(x) = 1$, we therefore find $\phi_1(x)$ and $\phi_2(x)$ when $n = 1$

$$a_1 = \frac{(x\phi_0, \phi_0)w}{\|\phi_0\|^2} \quad \text{since } w = \sqrt{x}$$

$$a_1 = \frac{(x,1)w}{\|1\|^2 w} = \frac{(x,1)\sqrt{x}}{1^2\sqrt{x}}$$

Inner product $(x,1) = x \cdot 1 = x$

e then solve for the third polynomial

$$\int_0^1 (x^3 - a_3x^2 + b_3x - c_3)\,dx = 0 \quad ---- \text{(i)}$$

$$\frac{x^4}{4} - \frac{a_3x^3}{3} + \frac{b_3x^2}{2} + c_3x \Big|_0^1 = 0$$

$$\Rightarrow \quad -\frac{a_3}{3} + \frac{b_3}{2} - c_3 = -\frac{1}{4} \quad ---- (*)$$

$$\int_0^1 (x^4 - a_3x^3 + b_3x^2 - c_3x)\,dx = 0 \quad ----- \text{(ii)}$$

$$\frac{x^5}{5} - \frac{a_3x^4}{4} + \frac{b_3x^3}{3} - \frac{c_3x^2}{2} \Big|_0^1 = 0$$

$$\Rightarrow \quad -\frac{a_3}{4} + \frac{b_3}{3} - \frac{c_3}{2} = -\frac{1}{5} \quad ---- (**)$$

$$\int_0^1 (x^5 - a_3x^4 + b_3x^3 - c_3x^2)\,dx = 0 \quad ----- \text{(iii)}$$

$$\frac{x^6}{6} - \frac{a_3x^5}{5} + \frac{b_3x^4}{4} - \frac{c_3x^3}{3} \Big|_0^1 = 0$$

$$\Rightarrow \quad -\frac{a_3}{5} + \frac{b_3}{4} - \frac{c_3}{3} = -\frac{1}{6} \quad ---- (***)$$

Using substitution and elimination method

From (*) : $\quad c_3 = -\frac{a_3}{3} + \frac{b_3}{2} + \frac{1}{4}$

Substitute for c_3 in (**) and (***)

$$-\frac{a_3}{4} + \frac{b_3}{3} - \left(\frac{-a_3/3 + b_3/2 + 1/4}{2}\right) = -\frac{1}{5}$$

$$-\frac{a_3}{5} + \frac{b_3}{4} - \left(\frac{-a_3/3 + b_3/2 + 1/4}{3}\right) = -\frac{1}{6}.$$

$$\left(-\frac{a_3}{4} + \frac{b_3}{3} - \left(\frac{-4a_3 + 6b_3 + 3}{24}\right) = -\frac{1}{5}\right) \times 120.$$

$$\left(-\frac{a_3}{5} + \frac{b_3}{4} - \left(\frac{-4a_3 + 6b_3 + 3}{36}\right) = -\frac{1}{6}\right) \times 180$$

$$-30a_3 + 40b_3 + 20a_3 - 30b_3 - 15 = -24$$

$$-36a_3 + 45b_3 + 20a_3 - 30b_3 - 15 = -30$$

$$-10a_3 + 10b_3 = -9 \quad --- \text{(i)}$$

$$-16a_3 + 15b_3 = -15 \quad --- \text{(ii)}$$

$$\therefore \quad a_1 = \frac{\int_0^1 x\sqrt{x}\,dx}{\int_0^1 \sqrt{x}\,dx} = \frac{\int_0^1 x^{3/2}\,dx}{\int_0^1 x^{1/2}\,dx}$$

$$a_1 = \frac{\frac{2}{5}x^{5/2}\Big|_0^1}{\frac{2}{3}x^{3/2}\Big|_0^1} = \frac{2/5}{2/3} = \frac{3}{5}$$

$$\therefore \phi_1(x) = x - a_1 = x - 3/5.$$

for $\phi_2(x)$

$$\phi_2(x) = x^2 - a_2 x + b_2$$

when $n=2$

$$a_2(x) = \frac{\int_0^1 (x\phi_1, \phi_1)w}{\int_0^1 \|\phi_1\|^2 w} \quad \text{but } \phi_1(x) = x - 3/5$$

$$\therefore \quad a_2(x) = \frac{\int_0^1 \left(x(x-3/5)(x-3/5)\right)\sqrt{x}\,dx}{\int_0^1 (x-3/5)^2\sqrt{x}\,dx}$$

2 ORTHOGONALITY OF THE CHEBYSHEV POLYNOMIAL.

The Chebyshev polynomial $\{T_n(x)\}$ are orthogonal on the interval $[-1,1]$ with respect to the weighted function $W(x) = \frac{1}{\sqrt{1-x^2}}$ i.e. $<$

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$$\langle T_i(x), T_j(x) \rangle w(x) \equiv \int_{-1}^{1} T_i(x) T_j(x) w(x) dx = \alpha_i \delta_{ij}$$

Suppose that,

$T_i(x) = T_n(x) = \cos(n \arccos\theta) = \cos n\theta$ where $\arccos\theta = \theta$

$T_j(x) = T_m(x) = \cos(m \arccos\theta) = \cos m\theta$

$$\langle T_i(x), T_j(x) \rangle w(x) = \int_{-1}^{1} \frac{T_n(x) T_m(x) w(x)}{\sqrt{1-x^2}} dx = \int_{-1}^{1} \frac{\cos(n\arccos\theta)\cos(m\arccos\theta)}{\sqrt{1-x^2}} dx$$

Introduce $\theta = \arccos x$

$d\theta = \frac{-dx}{\sqrt{1-x^2}}$ and the integral

$$-\int_{\pi}^{0} \cos(n\theta)\cos(m\theta)\, d\theta = \int_{0}^{\pi} \cos(n\theta)\cos(m\theta)\, d\theta$$

Recall: $\cos(n\theta)\cos(m\theta) = \frac{\cos(n+m)\theta + \cos(n-m)\theta}{2}$

$$= \int_{0}^{\pi} \frac{\cos(n+m)\theta + \cos(n-m)\theta}{2} d\theta$$

if $m \neq n$

$$= \frac{1}{2(n+m)} \sin(n+m)\theta + \frac{1}{2(n-m)} \sin(n-m)\theta \Big|_{0}^{\pi}$$

$$= \left[\frac{\sin(n+m)\pi}{2(n+m)} + \frac{\sin(n-m)\pi}{2(n-m)}\right] - \left[\frac{\sin(n+m)0}{2(n+m)} + \frac{\sin(n-m)0}{2(n-m)}\right] = 0$$

Hence,

$\langle T_i, T_j \rangle = 0$, Thus, it is orthogonal.

$n=1$ $\frac{\sin(2)\pi}{4} + \frac{\sin(0)\pi}{0}$

$m=1$

PROPERTIES OF CHEBYSHEV POLYNOMIAL (C.P.).

...ynomial $T_n(x)$ where $n \geq 1$ satisfies the following properties

...e C.P $T_n(x)$ satisfy the three term recurrence relation

$$T_{n+1}(x) = 2xT_n - T_{n-1}(x)$$

...e leading coefficient x^n of $T_n(x)$ is 2^{n-1} and $T_n(-x) = (-1)^n T_n(x)$

...REM.

...lynomial $\tilde{T}_n(x) = 2^{n-1} T_n(x)$ is the minimax approximation on $[-1,1]$ to the ... function by a monic polynomial of degree n and $\|\tilde{T}_n\| = 2^{1-n}$

...OF

...s suppose that there exist a monic polynomial P_n of degree n such that ...$| \leq 2^{1-n}$ $\forall$ $x \in [-1,1]$, and we will arrive at a contradiction.

...x_k', $K = 0,1,2,\ldots,n$ be the abscissas of the extreme values of the Chebyshev ...nomial of degree n. Because of property (ii) of this section we have

...$) < 2^{1-n} T_n(x_0')$, $P_n(x_1') > 2^{1-n} T_n(x_1')$, $P_n(x_2') > 2^{1-n} T_n(x_2')$, ----

...fore, the polynomial

$$Q(x) = P_n(x) - 2^{1-n} T_n(x)$$

...nges sign between each two consecutive extrema of $T_n(x)$. Thus, it changes ... n times. But this is not possible because $Q(x)$ is a polynomial of ...ee smaller than n (it is a subtraction of two monic polynomials of degree n).

...ARK.

...monic Chebyshev polynomial $\tilde{T}_n(x)$ is not the minimax approximation ...n of the zero function. The minimax approximation in P_n of the zero ...tion is the zero polynomial.

...THER PROPERTIES.

...elations with derivatives

$$\begin{cases} T_0(x) = T_1'(x) \\ T_1(x) = \frac{1}{4} T_2'(x) \\ T_n(x) = \frac{1}{2}\left(\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1}\right), \quad n \geq 2 \end{cases}$$

$$\ldots - x^2) T_n'(x) = n\left[xT_n(x) - T_{n+1}(x)\right] = n\left[T_{n-1}(x) - xT_n(x)\right]$$

...Multiplication relation.

$$2T_r(x)\, T_q(x) = T_{r+q}(x) + T_{|r-q|}(x)$$

with the particular case $q = 1$

$$2xT_r(x) = T_{r+1}(x) + T_{|r-1|}(x)$$

CHEBYSHEV POLYNOMIAL.

Chebyshev polynomial are used in many part of numerical analysis and more generally, in applied mathematics. For an integer $n \geq 0$, we define the function

$$T_n(x) = \cos(n\cos^{-1}x) \quad \text{--- --- ---} (4.0.1)$$

$$-1 \leq x \leq 1$$

This may not appear to be a polynomial but we will show that it is a polynomial of degree n. To simplify equation (4.0.1), we introduce

$$\theta = \cos^{-1}(x) \quad \text{--- --- ---} (4.0.2)$$

$$\text{or} \quad x = \cos\theta \quad \text{--- --- ---} (4.0.3)$$

$$0 \leq \theta \leq \pi$$

Then

$$T_n(x) = \cos(n\theta) \quad \text{--- --- ---} (4.0.4)$$

Hence, (4.0.4) is called recurrence relation

When $n=0$

$$T_0(x) = \cos(0\times\theta) = \cos 0 = 1$$

when $n=1$

$$T_1(x) = \cos(1\times\theta) = \cos\theta = x$$

when $n=2$

$$T_2(x) = \cos(2\times\theta) = \cos 2\theta$$

Recall that: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$= \cos^2\theta - (1-\cos^2\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\therefore T_2(x) = 2\cos^2\theta - 1$$

Since $\cos^2\theta = x^2$

$$T_2(x) = 2x^2 - 1$$

THE TRIPLE RECURSION RELATION OF CHEBYSHEV POLYNOMIAL.

Recall:

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$ ——— (4.1.1)

Let $n \geq 1$, apply the identities to obtain

$T_n = \cos(n\theta)$

$T_{n+1}(x) = \cos((n+1)\theta) = \cos(n\theta + \theta)$ ——

$T_{n+1}(x) = \cos(n\theta + \theta) = \cos n\theta \cos\theta - \sin n\theta \sin\theta$ ————— (4.1.2)

$T_{n-1}(x) = \cos(n\theta - \theta) = \cos n\theta \cos\theta + \sin n\theta \sin\theta$ ——— (4.1.3)

Adding (4.1.2) and (4.1.3)

$T_{n+1}(x) + T_{n-1}(x)$

$= \cos n\theta\cos\theta - \sin n\theta \sin\theta + \cos n\theta \cos\theta + \sin n\theta \sin\theta$

$= 2\cos n\theta \cos\theta$

Recall:

$x = \cos\theta$

$T_n = \cos(n\theta)$

$T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta \cos\theta = 2xT_n$

$\therefore T_{n+1}(x) + T_{n-1}(x) = 2xT_n$

$\Rightarrow T_{n+1}(x) = 2xT_n - T_{n-1}(x)$

For $n \geq 1$

$T_0(x) = 1, \quad T_1(x) = x$

$T_2(x) = 2xT_1(x) - T_0(x)$

$T_2(x) = 2x^2 - 1$

EXAMPLE 1.

Obtain the explicit expression for the first 10 chebyshev polynomial of the first kind i.e $-1 \leq x \leq 1$

$T_3(x) = 2xT_2(x) - T_1(x) = 2x(2x^2-1) - x$

$T_3(x) = 4x^3 - 3x$

$T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (2x^2-1)$

$T_4(x) = 8x^4 - 8x^2 + 1$

$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$

$T_5(x) = 16x^5 - 16x^3 + 2x - 4x^3 + 3x$

$T_5(x) = 16x^5 - 20x^3 + 5x$

$$T_6(x) = 2xT_5(x) - T_4(x)$$
$$= 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1)$$
$$T_6(x) = 32x^6 - 40x^4 + 18x^2 - 1$$

$$T_7(x) = 2xT_6 - T_5(x)$$
$$= 2x(32x^6 - 40x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x)$$
$$T_7(x) = 64x^7 - 96x^5 + 56x^3 - 7x$$

$$T_8(x) = 2xT_7(x) - T_6(x)$$
$$= 2x(64x^7 - 96x^5 + 56x^3 - 7x) - (32x^6 - 40x^4 + 18x^2 - 1)$$
$$T_8(x) = 128x^8 - 324x^6 + 152x^4 - 32x^2 + 1$$

$$T_9(x) = 2xT_8(x) - T_7(x)$$
$$= 2x(128x^8 - 324x^6 + 152x^4 - 32x^2 + 1) - (64x^7 - 96x^5 + 56x^3 - 7x)$$
$$T_9(x) = 256x^9 - 712x^7 + 398x^5 - 120x^3 + 9x$$

$$T_{10}(x) = 2xT_9(x) - T_8(x)$$
$$= 2x(256x^9 - 712x^7 + 398x^5 - 120x^3 + 9x) - (128x^8 - 324x^6 + 152x^4 - 32x^2 + 1)$$
$$T_{10}(x) = 512x^{10} - 1552x^8 + 1120x^6 - 392x^4 + 50x^2 + 1$$

SHIFTED CHEBYSHEV POLYNOMIALS.

.4 Shifted Chebyshev polynomial are also of interest when the range of independent variable is $[0,1]$ instead of $[-1,1]$.

The shifted C.P. of the first kind are

Recall:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$\text{let } x = \left(\frac{2x-a-b}{b-a}\right) = 2x-1 \quad \text{at } [0,1]$$

$T_{n+1}(x) = 2(2x-1)T_n(x) - T_{n-1}(x)$ is called the shifted C.P relation.

$T_0(x) = 1$ and $T_1(x) = 2x-1$

Explicit expression for the first six shifted Chebyshev Polynomial are

$$T_2(x) = 2(2x-1)(2x-1) - 1$$

$$= 8x^2 - 8x + 1$$

$$T_3(x) = 2(2x-1)T_2(x) - T_1(x)$$

$$= 2(2x-1)(8x^2-8x+1) - (2x-1)$$

$$T_3(x) = 32x^3 - 48x^2 + 18x - 1$$

$$T_4(x) = 2(2x-1)T_3(x) - T_2(x)$$

$$= 2(2x-1)(32x^3-48x^2+18x-1) - (8x^2-8x+1)$$

$$T_4(x) = 128x^4 - 256x^3 + 160x^2 - 2x + 1$$

$$T_5(x) = 2(2x-1)T_4(x) - T_3(x)$$

$$= 2(128x^4 - 256x^3 + 160x^2 - 2x + 1)(2x-1) - (32x^3 - 48x^2 + 18x - 1)$$

$$T_5(x) = 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1$$

ITERATIVE METHODS FOR SOLVING LINEAR EQUATIONS.

O
As a numerical technique, Guassian elimination method is rather unusual because it is direct i.e a solution is obtained in a simple application of Guassian elimination.

Once a solution has been obtained, Guassian elimination method offers no method of refinement.

Numerical techniques are commonly involved in iterative method for instance, in calculus, you probably studied Newton's method for approximating the zeros of a differential equation.

In this study, we will look into 2 iterative methods for approximating the solutions of n-linear equations in n-variables. For the purpose of convergence and divergence, the two (2) iterative methods are

(i) The Jacobi method

(ii) Guass-Siedel method.

1 THE JACOBI METHOD.

This method makes two assumptions that the system of given equation by

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ___ + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ___ + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ___ + a_{nn}x_n &= b_n
\end{aligned}$$

has a unique solution and that the coefficient of matrix A has no zero on its main diagonal entries has zero, then rows or columns must be interchanged to obtain a coefficient matrix that has no zero entries on the diagonal.

To begin the Jacobi method, solve the first equation for x_1, second equation for x_2, third equation for x_3 and so on as follows.

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \_\_\_ - a_{1n}x_n)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \_\_\_ - a_{2n}x_n).$$

$$x_3 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2 - \_\_\_ - a_{3n}x_n)$$

$$\vdots$$

$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \_\_\_ - a_{n,n-1}x_{n-1})$$

then make an initial approximation solution of $(x_1, x_2, x_3, ---, x_n)$.

NB: Every convergent functions are stable functions and all stable functions are bounded.

AMPLE 1.

...ing the Jacobi method to approximate the solutions of the following ...tem of equations.

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

$$x_1 = -1/5 + 2/5 x_2 - 3/5 x_3$$
$$x_2 = 2/9 + 3/9 x_1 - 1/9 x_3$$
$$x_3 = -3/7 + 2/7 x_1 - 1/7 x_2$$

With the initials $x_1 = x_2 = x_3 = 0$.

	N							
x	0	1	2	3	4	5	6	7
x_1	0.000	-0.200	0.146	0.192	0.181	0.185	0.186	0.186
x_2	0.000	0.222	0.202	0.325	0.332	0.329	0.331	0.331
x_3	0.000	-0.429	-0.517	-0.416	-0.421	-0.424	-0.423	-0.423

where N = Computational length.

...om the table, it is obvious that the system of equations converges at ...7, hence, the system of equations is stable and bounded.

XAMPLE 2.

GAUSS - SIEDEL METHOD.

2
This method holds if an $n \times n$ matrix A is strictly diagonally dominated

i.e $|a_{11}| > |a_{12}| + |a_{13}| + \dots + |a_{1n}|$

$|a_{22}| > |a_{21}| + |a_{23}| + \dots + |a_{2n}|$

$\vdots$

$|a_{nn}| > |a_{n1}| + |a_{n2}| + \dots + |a_{nn-1}|$

EXAMPLE

$5x_1 - 2x_2 + 3x_3 = -1$

$-3x_1 + 9x_2 + x_3 = 2$

$2x_1 - x_2 - 7x_3 = 3$

$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$

$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$

$x_3 = -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2$

N

x	0	1	2	3	4	5
x_1	0.000	-0.200	0.167	0.191	0.186	0.186
x_2	0.000	0.156	0.334	0.333	0.331	0.331
x_3	0.000	-0.508	-0.429	-0.422	-0.423	-0.423

where N = computational length.

Hence, the method converges at N=5 which is faster than Jacobi method. This method is a modification of Jacobi method.

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Test for convergence

$x^1 = Ax^0 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 11\begin{pmatrix} 0.818 \\ 1.000 \end{pmatrix} = \begin{pmatrix} 9 \\ 11 \end{pmatrix}$

$x^2 = Ax^1 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 9 \\ 11 \end{pmatrix} = 109\begin{pmatrix} 0.835 \\ 1.000 \end{pmatrix} = \begin{pmatrix} 91 \\ 109 \end{pmatrix}$

$x^3 = Ax^2 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 91 \\ 109 \end{pmatrix} = \begin{pmatrix} 909 \\ 1091 \end{pmatrix} = 1091\begin{pmatrix} 0.833 \\ 1.000 \end{pmatrix}$

$x^4 = Ax^3 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 909 \\ 1091 \end{pmatrix} = \begin{pmatrix} 9091 \\ 10909 \end{pmatrix} = 10909\begin{pmatrix} 0.833 \\ 1.000 \end{pmatrix}$

It converges at the 4th iteration.

EXAMPLE 2.

$$B = \begin{pmatrix} -4 & 10 \\ 7 & 5 \end{pmatrix}$$

$$|B - \lambda I| = \left|\begin{pmatrix} -4 & 10 \\ 7 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right|$$

$$= \begin{vmatrix} -4-\lambda & 10 \\ 7 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)(-4-\lambda) - 70$$

$$= \lambda^2 - \lambda - 20 - 70$$

$$= \lambda^2 - \lambda - 90$$

$x^1 = \begin{pmatrix} 0.500 \\ 1.000 \end{pmatrix}$, $x^2 = \begin{pmatrix} 0.941 \\ 1.000 \end{pmatrix}$, $x^3 = \begin{pmatrix} 0.812 \\ 1.000 \end{pmatrix}$, ---

--- $x^{66} = \begin{pmatrix} 0.715 \\ 1.000 \end{pmatrix}$, $x^{67} = \begin{pmatrix} 0.714 \\ 1.000 \end{pmatrix}$, $x^{68} = \begin{pmatrix} 0.714 \\ 1.000 \end{pmatrix}$

NOTE: The first example converges easily because the ratio of eigen value is small (0.1) while example 2 converges slowly because the ratio of the eigen values is big i.e close to 1.

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6.0 NUMERICAL SOLUTION OF EIGEN-VALUE & EIGEN-VECTOR.

6.1 DEFINITION

Let $\lambda_1, \lambda_2, ----, \lambda_n$ be the eigen-values of an $n \times n$ matrix A. λ_1 is called the dominant eigen value of A if

$$|\lambda_1| > |\lambda_i| \; ; \; i = 2, 3, 4, ---, n$$

The corresponding eigen-vector for the dominant eigen value is called DOMINANT EIGEN VECTOR.

e.g $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(-1-\lambda) = 0$$

$$-1 - \lambda + \lambda + \lambda^2 = 0.$$

$$\lambda^2 = 1$$

$$\Rightarrow \lambda = \pm\sqrt{1}$$

$$\lambda = \pm 1$$

$$|\lambda_1| = |\lambda_2|$$

$\therefore$ no dominant eigen value.

eg Suppose, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\lambda_1 = 2, \; \lambda_2 = 2, \text{ and } \lambda_3 = 1$$

$$|\lambda_1| = |\lambda_2|$$

hence, no dominant eigen value hence, no dominant eigen vector.

$C = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \Rightarrow |C - I\lambda| = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\lambda_1 = -2 \text{ or } \lambda_2 = -1$$

$$|\lambda_1| = 2 > |\lambda_2| = 1$$

$\Rightarrow \begin{pmatrix} 2-\lambda & -12-0 \\ 1-0 & -5-\lambda \end{pmatrix} = (2-\lambda)(-5-\lambda) = 0$

hence, dominant eigen value

Similarly, the dominant eigen vector for dominant eigen values $\lambda_1 = 2$

$$X = \begin{bmatrix} 2-\lambda_1 & -12 \\ 1 & -5-\lambda_1 \end{bmatrix} = \begin{bmatrix} 4 & -12 \\ 1 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$4x_1 - 12x_2 = 0 \text{ and } x_1 - 3x_2 = 0$$

$$\therefore x_1 = 3x_2$$

$$X = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \; ; \; t \neq 0.$$

$\lambda^2 + 5\lambda$ ... $\lambda + 5$ ... -2

THE POWER METHOD.

.2 Firstly, we assume that a matrix A has a dominant eigen value with corresponding eigen vector.
Then we choose an initial approximation x^0 of one of the dominant eigen vectors of A. This initial approximation must be a non-zero vector in $\mathbb{R}^n$.
Finally, we form a sequence as follows.

$$x^1 = Ax^0$$

$$x^2 = AAx^0 = A^2x^0$$

$$x^3 = AAAx^0 = A^3x^0$$

$$\vdots$$

$$x^k = Ax^{k-1} = A(x^{k-1}x^0) = A^kx^0$$

For large power of k and by properly scaling this sequence, we will see that we will obtain a good approximation of a dominant eigen vector of A.

.3 THEOREM

If x is an eigen vector of a matrix A, then its corresponding eigen value is given by this theorem as

$$\lambda = \frac{Ax \cdot x}{x \cdot x}$$

PROOF

Since x is an eigen vector of A we know that $Ax = \lambda x$ and we compute that

$$\frac{Ax \cdot x}{x \cdot x} = \frac{\lambda x \cdot x}{x \cdot x} = \frac{\lambda(x \cdot x)}{(x \cdot x)} = \lambda$$

EXAMPLE 1.

Using power method, consider a matrix A and complete the 6 iteration to obtain an approximate dominant eigen vector and corresponding dominant eigen value.

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \quad \lambda_1 = -2 \qquad x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = t\begin{bmatrix} 3 \\ 1 \end{bmatrix}; \; t \neq 0$$

$$x^1 = Ax^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ -4 \end{pmatrix} = -4\begin{pmatrix} 2.50 \\ 1.00 \end{pmatrix} \quad (2\text{d.p.})$$

$$x^2 = A^2x^0 = A \cdot Ax^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} -10 \\ -4 \end{pmatrix} = \begin{pmatrix} 28 \\ 10 \end{pmatrix} = 10\begin{pmatrix} 2.80 \\ 1.00 \end{pmatrix}$$

$$x^3 = A^3x^0 = A \cdot A^2x^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 28 \\ 10 \end{pmatrix} = \begin{pmatrix} -64 \\ -22 \end{pmatrix} = -22\begin{pmatrix} 2.91 \\ 1.00 \end{pmatrix}$$

$$x^4 = A^4x^0 = A\cdot A^3x^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} -64 \\ -22 \end{pmatrix} = \begin{pmatrix} 136 \\ 46 \end{pmatrix} = 46\begin{pmatrix} 2.96 \\ 1.00 \end{pmatrix}$$

$$x^5 = A^5x^0 = A\cdot A^4x^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 136 \\ 46 \end{pmatrix} = \begin{pmatrix} -280 \\ -94 \end{pmatrix} = -94\begin{pmatrix} 2.98 \\ 1.00 \end{pmatrix}$$

$$x^6 = A^6x^0 = A\cdot A^5x^0 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} -280 \\ -94 \end{pmatrix} = \begin{pmatrix} 568 \\ 190 \end{pmatrix} = 190\begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix}$$

$\therefore \begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix}$ is the dominant eigen vector as compared to $t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix} = x$ is the approximate dominant eigen vector.

To obtain the corresponding eigen value, we have

$$\lambda = \frac{Ax\cdot x}{x\cdot x}$$

$$Ax = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix} = \begin{pmatrix} -6.02 \\ -2.00 \end{pmatrix}$$

$$Ax\cdot x = \begin{pmatrix} -6.02 \\ -2.00 \end{pmatrix}\begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix}$$

$$= (-6.02 \times 2.99) + (-2.00 \times 1.00) = -20.00$$

$$x\cdot x = \begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix}\begin{pmatrix} 2.99 \\ 1.00 \end{pmatrix} = 9.94$$

$$\therefore \lambda = \frac{-20.00}{9.94} = \underline{\underline{-2.01}}$$

$\therefore \lambda = -2.01$ is the dominant eigen value.

EXAMPLE 2.

$$P = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix} \qquad x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x^1 = Px^0 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5\begin{pmatrix} 0.60 \\ 0.20 \\ 1.00 \end{pmatrix}$$

$$x^2 = Px^1 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 3.00 \\ 1.00 \\ 5.00 \end{pmatrix} = \begin{pmatrix} 5.00 \\ 5.00 \\ 11.00 \end{pmatrix} = 11\begin{pmatrix} 0.45 \\ 0.45 \\ 1.00 \end{pmatrix}$$

$$x^3 = Px^2 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 5.00 \\ 5.00 \\ 11.00 \end{pmatrix} = \begin{pmatrix} 15.00 \\ 17.00 \\ 31.00 \end{pmatrix} = 31\begin{pmatrix} 0.48 \\ 0.55 \\ 1.00 \end{pmatrix}$$

DG

$$x^4 = Px^3 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 15 \\ 17 \\ 31 \end{pmatrix} = \begin{pmatrix} 49 \\ 49 \\ 97 \end{pmatrix} = 97\begin{pmatrix} 0.51 \\ 0.51 \\ 1.00 \end{pmatrix}$$

$$x^5 = Px^4 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 49 \\ 49 \\ 97 \end{pmatrix} = \begin{pmatrix} 147 \\ 145 \\ 293 \end{pmatrix} = 293\begin{pmatrix} 0.50 \\ 0.49 \\ 1.00 \end{pmatrix}$$

$$x^6 = Px^5 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 147 \\ 145 \\ 293 \end{pmatrix} = \begin{pmatrix} 437 \\ 437 \\ 875 \end{pmatrix} = 875\begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix}$$

$$x^7 = Px^6 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 437 \\ 437 \\ 875 \end{pmatrix} = \begin{pmatrix} 1311 \\ 1313 \\ 2623 \end{pmatrix} = 2623\begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix}$$

$$\lambda = \frac{Px \cdot x}{x \cdot x}$$

$$Px = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix} = \begin{pmatrix} 1.5 \\ 1.5 \\ 3 \end{pmatrix}$$

$$Px \cdot x = \begin{pmatrix} 1.5 \\ 1.5 \\ 3 \end{pmatrix}\begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix} = 4.5$$

$$x \cdot x = \begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix}\begin{pmatrix} 0.50 \\ 0.50 \\ 1.00 \end{pmatrix} = 1.5$$

$$\therefore \lambda = \frac{4.5}{1.5} = \underline{\underline{3}}$$

The dominant eigen value $\lambda = 3$.

CONVERGENCE OF POWER METHOD.

6.4 If A is an $n\times n$ diagonal matrix with a dominant eigen value, $\exists$ a non-zero vector x^0 $\ni$ the sequence of vector is given by $Ax^0, A^2x^0, A^3x^0, A^4x^0, ---, A^nx^0$ approaches a multiple of dominant eigen vector of A.

PROOF
If A is a diagonalized matrix, which has n - linearly independent eigen vectors $x^1, x^2, x^3, ---, x^n$ with corresponding dominant eigen values of $\lambda_1, \lambda_2, ---, \lambda_n$ we assume λ_1 as dominant eigen value with corresponding eigen vector x^1. Because the n eigen vectors $x^1, x^2, ---, x^n$ are linearly independent, they must form basis for $\mathbb{R}^n$ for initial approximation x^0.

$$x^0 = c_1x^1 + c_2x^2 + --- + c_nx^n \quad ——— (6.4.1)$$

If $c_1 = 0$, the power method may not converge and a different x^0 must be used as the initial approximation. Multiply both side of (6.4.1) with A

$$x^1 = Ax^0 = A\left(c_1x^1 + c_2x^2 + c_3x^3 + --- + c_nx^n\right)$$
$$= c_1(Ax^1) + c_2(Ax^2) + c_3(Ax^3) + --- + c_n(Ax^n)$$
$$= c_1(\lambda_1x^1) + c_2(\lambda_2x^2) + c_3(\lambda_3x^3) + --- + c_n(\lambda_nx^n)$$

Similarly,

$$x^2 = A^2x^0 = c_1(\lambda_1^2x^1) + c_2(\lambda_2^2x^2) + c_3(\lambda_3^2x^3) + --- + c_n(\lambda_n^k x^n)$$

$$x^k = A^kx^0 = \lambda_1^k\left[c_1x^1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k x^2 + c_3\left(\frac{\lambda_3}{\lambda_1}\right)^k x^3 + ——— + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k x^k\right]$$

If $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, ---, \frac{\lambda_n}{\lambda_1}$ are less than 1 in absolute value, therefore, each of the factors $\left(\frac{\lambda_2}{\lambda_1}\right)^k, \left(\frac{\lambda_3}{\lambda_1}\right)^k, ---, \left(\frac{\lambda_n}{\lambda_1}\right)^k$ must approach 0 as $k \to \infty$.

Thus, if $\frac{|\lambda_2|}{|\lambda_1|}$ is small, it converges faster.

if $\frac{|\lambda_2|}{|\lambda_1|}$ is close to 1, converges slowly.

EXAMPLE 1.

$$A = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix} \qquad \lambda_1 = 10 \text{ and } \lambda_2 = -1$$

$$\frac{|\lambda_2|}{|\lambda_1|} = \frac{1}{10} = 0.1$$

6.5 THE INVERSE POWER METHOD.

To apply this method, the following shall be considered.

(i) An eigen value other than the dominant one.

(ii) To derive the inverse power method, we need

The relationship between the eigen values of a matrix A to a class of matrices constructed from A.

6.6 THEOREM.

B is a polynomial of matrix A. Let A be an $n \times n$ matrix with eigen values $\lambda_1, \lambda_2, \lambda_3, ---, \lambda_n$ and the associated eigenvectors $V_1, V_2, V_3, ---, V_n$.

(i) If $B = a_0 I + a_1 A + a_2 A^2 + --- + a_m A^m = P(A)$, where P is the polynomial -

$P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ---- + a_m x^m$, then the eigenvalues of B are

$P(\lambda_1), P(\lambda_2), P(\lambda_3), ----, P(\lambda_n)$ with associated eigenvectors $V_1, V_2, V_3, ---, V_n$.

(ii) If A is non-singular, then A^{-1} has eigen values $1/\lambda_1, 1/\lambda_2, 1/\lambda_3, ----, 1/\lambda_n$ with associated eigenvectors $V_1, V_2, V_3, ---, V_n$.

PROOF

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, ---, \lambda_n$ and associated eigenvectors $V_1, V_2, ---, V_n$.

⇒ PART I: Note that for any positive integer k.

$$A^k V_i = A^{k-1}(AV_i) = \lambda_i A^{k-1} V_i$$

$$= \lambda_i A^{k-2}(AV_i) = \lambda_i^2 A^{k-2} V_i$$

$$= - - - -$$

$$= \lambda_i^{k-1}(AV_i) = (\lambda_i^k V_i).$$

Now let $B = a_0 I + a_1 A + a_2 A^2 + --- + a_m A^m = P(A)$ where P is the polynomial.

$P(x) = a_0 + a_1 x + a_2 x^2 + ---+ a_m x^m$, then for each $i = 1, 2, 3, --, n$.

$BV_i = V_i(a_0 I + a_1 A + a_2 A^2 + --- + a_m A^m) = a_0 V_i + a_1 AV_i + a_2 A^2 V_i + --- + a_m A^m V_i$

$= a_0 V_i + a_1 \lambda_i V_i + a_2 \lambda_i^2 V_i + --- + a_m \lambda_i^m V_i = P(\lambda_i) V_i$

Hence, the eigen value of B are $P(\lambda_1), P(\lambda_2), ---, P(\lambda_n)$ with associated eigenvectors $V_1, V_2, ---, V$

⇒ PART II: Suppose A is non-singular. Since V_i is an eigen vector associated with the eigenvalues λ_i, it follows that $AV_i \equiv \lambda_i V_i$

Premultiplying this equation by $(1/\lambda_i)A^{-1}$ yields

$$\left(\frac{1}{\lambda_i}\right) A^{-1}(AV_i) = \frac{1}{\lambda_i} A^{-1}(\lambda_i V_i)$$

$$A^{-1} \cdot AV_i \cdot 1/\lambda_i = 1/\lambda_i A^{-1} \lambda_i V_i$$

$$1/\lambda_i V_i = A^{-1} V_i$$

Therefore, for each $i = 1, 2, 3, ----, n$, $1/\lambda_i$ is an eigenvalue of A^{-1}, with associated eigenvectors V_i.

7.0 NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATION.

Ordinary Differential Equation arises frequently in Physics, Chemistry, Biology, Engineering, Economics etc. Most of the solution of the equation may not be found in analytical way, hence, there is a need for numerical approach.

A general form of Ordinary Differential Equation (O.D.E)

$$y^{n} = f(x, y', y'', y''', \dots, y^{n-1} \quad \text{------} \quad (7.0.1)$$

Where y is a function of a single variable x and n contains in positive integers ($n \in Z^{+}$) and is the order of the Ordinary Differential Equation (O.D.E. The numerical method involves estimating the value of x for discrete values. So, if we want to integrate the above (7.0.1) from $x_1 = a$ and $x_2 = b$, we divide $[x_1, x_2] = [a, b]$ into numbers of some intervals each of which is the length h so that $x_i = x_0 + ih$, $i = 1, 2, 3, 4, \dots, n$.

Here $x_0 = a$ and $x_n = b$.

We obtain a numerical solution

$$(x_i, y_i) \; ; \; i = (0, n)$$

and y_i is the estimate of exact solution (analytical solution).

Consider the first-order differential equation

$$y' = f(x, y) \quad \text{------} \quad (7.0.2)$$

Subject to $y(x_0) = y_0$ ------ (7.0.3)

Using Taylor's series to get an approximate solution from y_1 to y_n

$$y_1 = y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \frac{h^4 y_0^{IV}}{4!} + \dots + \frac{h^n y_0^n}{n!}$$

$$y_2 = y_1 + hy_1' + \frac{h^2 y_1''}{2!} + \frac{h^3 y_1'''}{3!} + \frac{h^4 y_1^{IV}}{4!} + \dots + \frac{h^n y_1^n}{n!}$$

$$y_3 = y_2 + hy_2' + \frac{h^2 y_2''}{2!} + \frac{h^3 y_2'''}{3!} + \frac{h^4 y_2^{IV}}{4!} + \dots + \frac{h^n y_2^n}{n!}$$

$$\vdots$$

$$y_n = y_{n-1} + hy_{n-1}' + \frac{h^2 y_{n-1}''}{2!} + \frac{h^3 y_{n-1}'''}{3!} + \frac{h^4 y_{n-1}^{IV}}{4!} + \dots + \frac{h^n y_{n-1}^n}{n!}$$

7.1 SOLUTION TECHNIQUES.

7.1.1 EULER'S METHOD.

This method is used to solve first order differential equation. Suppose we want to find the numerical solution of

$$y' = f(x,y) \quad \text{------} \quad (7.1.1.1)$$

Subject to initial condition $y(x_0) = y_0$

From Taylor's series ($y_0' = f(x_0, y_0)$)

$$y_1 = y_0 + hy_0'$$

$$\therefore \quad y_1 = y_0 + hf(x_0, y_0) \quad \text{-----} \quad (7.1.1.2)$$

Generally, Euler's method is

$$y_{K+1} = y_K + hf(x_K, y_K) \quad \text{-----} \quad (7.1.1.3)$$

Example 1.

Solve $y' = x^2 y$ in $[0, 1]$ choosing $h = 0.1$

Subject to $y(0) = 1$ using Euler's method.

Solution

To find the exact solution

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln y = \frac{x^3}{3} + c$$

Exact solution, $y(x) = e^{x^3/3}$

Using Euler's method

$$y_{K+1} = y_K + hf(x_K, y_K)$$

$$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$$

From $y(0) = 1 \Rightarrow x_0 = 0$ and $y_0 = 1$

$$f(x_0, y_0) = f(0, 1)$$

$$f(x_0, y_0) = x^2 y = (0)^2 \cdot 1 = 0$$

$$\therefore y_1 = 1 + 0.1(0) = 1.0000$$

$\Rightarrow y_2 = y_1 + hf(x_1, y_1)$

Since $h = 0.1 \Rightarrow x_i$ is increasing at a step length of 0.1

$\therefore f(x_1, y_1) = f(0.1, 1.000) = x^2 y = (0.1)^2 \cdot 1 = 0.01$

$y_2 = 1.000 + 0.1(0.01) = 1.001$

$\Rightarrow y_3 = y_2 + hf(x_2, y_2)$

$f(x_2, y_2) = f(0.2, 1.001) = x^2 y = (0.2)^2 \times 1.001 = 0.04004$

$y_3 = 1.001 + 0.1(0.04004) = 1.00500$

$\Rightarrow y_4 = y_3 + hf(x_3, y_3)$

$f(x_3, y_3) = f(0.3, 1.00500) = x^2 y = (0.3)^2 \times 1.00500 = 0.09045$

$y_4 = 1.00500 + 0.1(0.09045) = 1.01405$

$\Rightarrow y_5 = y_4 + hf(x_4, y_4)$

$f(x_4, y_4) = f(0.4, 1.01405) = x^2 y = (0.4)^2 \times 1.01405 = 0.16225$

$y_5 = 1.01405 + 0.1(0.16225) = 1.030275$

$\Rightarrow y_6 = y_5 + hf(x_5, y_5)$

$f(x_5, y_5) = f(0.5, 1.030275) = x^2 y = (0.5)^2 \times 1.030275 = 0.257569$

$y_6 = 1.030275 + 0.1(0.257569) = 1.287844$

$\Rightarrow y_7 = y_6 + hf(x_6, y_6)$

$f(x_6, y_6) = f(0.6, 1.287844) = x^2 y = (0.6)^2 \times 1.287844 = 0.463624$

$y_7 = 1.287844 + 0.1(0.463624) = 1.334206$

$\Rightarrow y_8 = y_7 + hf(x_7, y_7)$

$f(x_7, y_7) = f(0.7, 1.334206) = x^2 y$

$= (0.7)^2 \times 1.334206 = 0.653761$

$y_8 = 1.334206 + 0.1(0.653761) = 1.399582$

$\Rightarrow y_9 = y_8 + hf(x_8, y_8)$

$f(x_8, y_8) = f(0.8, 1.399582) = x^2 y$

$= (0.8)^2 \times 1.399582 = 0.895733$

$y_9 = 1.399582 + 0.1(0.895733) = 1.489155$

$\Rightarrow y_{10} = y_9 + hf(x_9, y_9)$

$f(x_9, y_9) = f(0.9, 1.489155) = x^2 y$

$= (0.9)^2 \times 1.489155 = 1.206216$

$y_{10} = 1.489155 + 0.1(1.206216) = 1.609777$

$\Rightarrow y_{11} = y_{10} + hf(x_{10}, y_{10})$

$f(x_{10}, y_{10}) = f(1.0, 1.609777) = x^2 y$

$= (1.0)^2 \times 1.609777 = 1.609777$

$y_{11} = 1.609777 + 0.1(1.609777) = 1.770755$

x_i	Euler	Exact $(e^{x^3/3})$
0	1.000000	1.000000
0.1	1.001000	1.000330
0.2	1.005000	1.002670
0.3	1.014050	1.009400
0.4	1.030275	1.021560
0.5	1.287844	1.042550
0.6	1.334206	1.07466
0.7	1.399582	1.121130
0.8	1.489152	1.186100
0.9	1.609777	1.275070
1.0	1.770755	1.395610

NOTE:

It is observed that the Euler's method is falling behind exact solution which leads to propagation of error.

There fore, Euler's method is rarely used in solving Ordinary Differential Equation (O.D.E) in applied engineering and sciences.

However, it serves as a basis for other improved techniques.

7.1.2 TAYLOR'S METHOD.

Example 1.

$y' = x + y$ taking $h = 0.1$ subject to $y(1) = 0$

Exact solution, $y(x) = -x - 1 + 2e^{x-1}$

Obtain $y(1.1)$ and $y(1.2)$ by Taylor's series method

$$y_1 = y(1.1) = y_0 + hy_0' + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \cdots$$

$y(1) = 0 \Rightarrow x_0 = 1$ and $y_0 = 0$

$y_0' = x_0 + y_0 = 1 + 0 = 1$ $\Rightarrow x_0 + y_0 =$

$y_0'' = 1 + y_0' = 1 + 1 = 2$

$y_0''' = y_0'' = 2$

$y_0^{IV} = y_0''' = 2$

$h = 0.1$, $x_0 = 1$, $y_0 = 0$, $x_1 = 1.1$ $\Rightarrow x_1 = x_0 + h$

$$\therefore y_1 = y(1.1) = 0 + 0.1(1) + \frac{(0.1)^2 \cdot 2}{2!} + \frac{(0.1)^3 \cdot 2}{3!} + \frac{(0.1)^4 \cdot 2}{4!}$$

$y(1.1) = 0.11034167$

Exact, $y(1.1) = -x_1 - 1 + 2e^{x_1 - 1} = -1.1 - 1 + 2e^{1.1-1} = 0.11034184$

$$y_2 = y(1.2) = y_1 + hy_1' + \frac{h^2 y_1''}{2!} + \frac{h^3 y_1'''}{3!} + \frac{h^4 y_1^{IV}}{4!}$$

$y_1' = x_1 + y_1 = 1.1 + 0.11034167 = 1.21034167$

$y_1'' = 1 + y_1' = 1 + 1.21034167 = 2.21034167$

$y_1''' = y_1'' = 2.21034167$

$y_1^{IV} = y_1''' = 2.21034167$

$$\therefore y_2 = y(1.2) = 0.11034167 + 0.1(1.21034167) + \frac{(0.1)^2(2.21034167)}{2!} + \frac{(0.1)^3(2.21034167)}{3!} + \frac{(0.1)^4(2.21034167)}{4!}$$

$\therefore y(1.2) = 0.24280515$

Exact, $y(1.2) = -x_2 - 1 + 2e^{x_2 - 1} = -1.2 - 1 + 2e^{1.2-1} = 0.2428552$

NOTE: The Taylor's series method is an improvement and the originator of Euler's method

7.1.3. RUNGE - KUTTA METHOD.

This is a numerical method without any Taylor's series approach. Suppose we want to find the numerical solution of the form

$$y' = f(x,y) \quad \text{------} \quad (7.1.3.1)$$

Subject to $y(x_0) = y_0$ ------ (7.1.3.2)

To obtain k_1, k_2, k_3 and k_4 respectively

To obtain second order

$$k_1 = hf(x_0, y_0) \quad \text{----} \quad (7.1.3.3)$$

$$k_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_1) \quad \text{----} \quad (7.1.3.4)$$

hence,

$$\Delta y = k_2 \quad \text{------} \quad (7.1.3.5)$$

$$h = \Delta x \quad \text{-----} \quad (7.1.3.6)$$

The above algorithm is called second order Runge-Kutta method.

To obtain third order

$$k_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}k_2) \quad \text{------} \quad (7.1.3.7)$$

here $\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$ -------- (7.1.3.8)

(7.1.3.8) is called third order Runge-Kutta method.

To obtain fourth order

$$k_4 = hf(x_0 + h, y_0 + k_3) \quad \text{-----} \quad (7.1.3.9)$$

here $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ ----- (7.1.3.10)

(7.1.3.10) is called fourth order Runge-Kutta method.

The above 2nd, 3rd and 4th order are at the point (x_0, y_0)

At point (x_1, y_1)

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_1 + \tfrac{1}{2}h, y_1 + \tfrac{1}{2}k_2)$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

At point (x_n, y_n)

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$

$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$

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$= -0.2 - 1 + 2e^{0.2} = 1.242806$

NOTE:

$$K_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}K_1) = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}(hf(x_n, y_n)))$$

$$\equiv \quad y_1 = y_0 + h(f(x_0 + h/2, y_0 + h/2 f(x_0, y_0)))$$

This is equivalent to modified Euler's method, hence Runge-Kutta method of second order is nothing but modified Euler's method.

Example 2.

$y' = x + y$ subject to $y(0) = 1$ i.e $x_0 = 0$ and $y_0 = 1$

Exact solution, $y(x) = -x - 1 + 2e^{x}$

Using fourth order Runge Kutta method obtain $f(0.1)$ and $f(0.2)$ at $h = 0.1$

Solution

Recall the 4th order Runge-Kutta method is given as

$$\Delta y = \tfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$y' = x + y$

$\Rightarrow K_1 = hf(x_0, y_0)$

$= 0.1 \times f(0,1) = 0.1 \times (0+1) = 0.1 \times 1$

$K_1 = 0.1$

$\Rightarrow K_2 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}K_1)$

$x_0 + \tfrac{1}{2}h = 0 + \frac{0.1}{2} = 0.05$

$y_0 + \tfrac{1}{2}K_1 = 1 + \frac{0.1}{2} = 1.05$

$K_2 = hf(0.05, 1.05) = 0.1 \times (0.05 + 1.05) = 0.1 \times 1.10$

$K_2 = 0.11$

$\Rightarrow K_3 = hf(x_0 + \tfrac{1}{2}h, y_0 + \tfrac{1}{2}K_2)$

$x_0 + \tfrac{1}{2}h = 0.05$

$y_0 + \tfrac{1}{2}K_2 = 1 + \frac{0.11}{2} = 1.055$

$\therefore K_3 = hf(0.05, 1.055)$

$= 0.1 \times (0.05 + 1.055) = 0.1 \times 1.105$

$K_3 = 0.1105$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3)$

$x_0 + h = 0 + 0.1 = 0.1$

$y_0 + K_3 = 1 + 0.1105 = 1.1105$

$K_4 = 0.1 \times f(0.1, 1.1105) = 0.1 \times (0.1 + 1.1105)$

$\therefore \Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$= \frac{1}{6}(0.1 + 2(0.11) + 2(0.1105) + 0.12105)$

$\Delta y = 0.110342$

$\therefore y(0.1) = y_0 + \Delta y = 1 + 0.110342 = 1.110342$

Exact solution, $y(0.1) = -x-1+2e^x = -0.1-1+2e^{0.1} = 1.110342$

At point $(x_1, y_1) \equiv (0.1, 1.110342)$

$\Rightarrow k_1 = hf(x_1, y_1)$

$k_1 = 0.1 \times f(0.1, 1.110342) = 0.1 \times (0.1 + 1.110342)$

$k_1 = 0.1(1.210342) = 0.1210342$

$\Rightarrow k_2 = hf(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_1)$

$x_1 + \frac{1}{2}h = 0.1 + \frac{0.1}{2} = 0.15$

$y_1 + \frac{1}{2}k_1 = 1.110342 + \frac{0.1210342}{2} = 1.170859$

$\therefore k_2 = hf(0.15, 1.170859)$

$= 0.1 \times (0.15 + 1.170859) = 0.1 \times (1.320859)$

$k_2 = 0.1320859$

$\Rightarrow k_3 = hf(x_1+\frac{1}{2}h, y_1+\frac{1}{2}k_2)$

$x_1 + \frac{1}{2}h = 0.1 + \frac{0.1}{2} = 0.15$

$y_1 + \frac{1}{2}k_2 = 1.110342 + \frac{0.1320859}{2} = 1.176358$

$k_3 = 0.1 \times f(0.15, 1.176358) = 0.1 \times (0.15 + 1.176358)$

$\therefore k_3 = 0.132639$

$\Rightarrow k_4 = hf(x_1+h, y_1+k_3)$

$x_1 + h = 0.1 + 0.1 = 0.2$

$y_1 + k_3 = 1.110342 + 0.132639 = 1.242981$

$\therefore k_4 = 0.1 \times f(0.2, 1.242981) = 0.1 \times (0.2 + 1.242981)$

$k_4 = 0.144298$

$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.1210342 + 2(0.132086) + 2(0.132639) + 0.144298)$

$\Delta y = 0.132464$

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$\therefore y(0.2) = y_1 + \Delta y = 1.110342 + 0.132464 = 1.242806$

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Exact solution, $y(0.2) = -x_2 - 1 + 2e^{x_2} = -0.2 - 1 + 2e^{0.2} = 1.242806$